Multiport conversions between S, Z, Y, h, ABCD, and T parameters

T. Reveyrand
XLIM-CNRS
Limoges, France
tibo@xlim.fr

***Abstract*—This paper presents main formulas to convert S, Z, Y, h, ABCD and T parameters of multiport circuits. Formulas are presented in matrix form, and some restrictions on unbalanced systems for cascade parameters are discussed. Those expressions are prime of importance in order to analyze and optimize multiport systems without any simulation software.**

***Index Terms*—Multiport, Embedding, De-embedding, S-parameters, Calibration, Education.**

I. Introduction

Most microwave engineers have already developed some tools to simplify their tasks in high level language such as Python, Matlab, Scilab. Regarding passive device embedding and de-embedding procedures, for example, most of their work is based on the conversion between S (scattering), Z (impedance), Y (admittance), h (hybrid), ABCD (chain) and T (chain transfert) parameters limited to only 2-ports devices as available in [1]. These days, the need for multi-port conversion tools is illustrated by the growth of multiple input / output devices for RF and microwave applications such as dual-input power amplifier, multiport combiners in Doherty or Outphasing power amplifiers, multiport nonlinear transistor models or MIMO systems.

This paper starts from the reference paper regarding circuit-level matrix conversion written by Dean A. Frickey [1] and the discuss regarding the use of a complex impedance as reference for S-parameters [2], and proposes an extension to multiport analysis. Formulas, presented here, will help engineers to develop design optimization methods without the need of a linear S-parameter simulation.

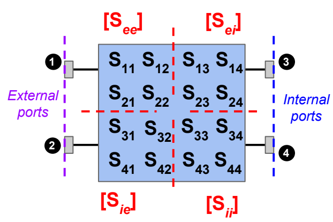

Fig. 1. Example of partitioning on a balanced (same number of port on the external and internal reference planes) multiport. The terminology ”external” and ”internal” comes from embedding /de-embedding problems where the device of interest is connected to the internal port.

The multiport approach, initially presented in [3], and well detailed in [4], consists on considering the S-parameter matrix of the device of interest as a partition of 4 sub-matrix. Ports are divided in 2 groups, often named external and internal ports according to embedding problems. Therefore, the 2-port analysis is easily extended to multiport purpose. Figure 1 illustrates a multiport S-matrix example and how the matrix is partitioned according to the external and internal ports considerations. This kind of partitioning can be directly applied to Z, Y, h, ABCD and T matrices as well.

The power-waves in use for S-parameters matrices are defined in [5] as:

$$\left\{ \begin{array}{ll} a_i & = \frac{1}{2.\sqrt{|\Re\{Z_i\}|}}.(V_i + Z_i.I_i) \\ b_i & = \frac{1}{2.\sqrt{|\Re\{Z_i\}|}}.(V_i - Z_i^*.I_i) \end{array} \right. \tag{1}$$

where a_i and b_i are respectively the incident and reflected power waves, V_i and I_i the voltage and currents and Z_i the reference impedance (normalization impedance) at port i. By substituting a_i and b_i from (1) in multiport circuit parameters matrices, generalized multiport conversions can be easily demonstrated. Results are presented in this paper.

II. Multiport Parameters Definition

Electrical parameters matrices, already defined in [1] can be expanded to multiport according to the partitioned sub-matrices defined in the introduction such as:

$$\begin{pmatrix} (V_e) \\ (V_i) \end{pmatrix} = \begin{bmatrix} [Z_{ee}] & [Z_{ei}] \\ [Z_{ie}] & [Z_{ii}] \end{bmatrix} \cdot \begin{pmatrix} (I_e) \\ (I_i) \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} (I_e) \\ (I_i) \end{pmatrix} = \begin{bmatrix} [Y_{ee}] & [Y_{ei}] \\ [Y_{ie}] & [Y_{ii}] \end{bmatrix} \cdot \begin{pmatrix} (V_e) \\ (V_i) \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} (V_e) \\ (I_i) \end{pmatrix} = \begin{bmatrix} [h_{ee}] & [h_{ei}] \\ [h_{ie}] & [h_{ii}] \end{bmatrix} \cdot \begin{pmatrix} (I_e) \\ (V_i) \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} (V_e) \\ (I_e) \end{pmatrix} = \begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \cdot \begin{pmatrix} (V_i) \\ (-I_i) \end{pmatrix} \tag{5}$$

Furthermore, multiport S-parameters may be presented as the following set of submatrices :

$$\begin{pmatrix} (b_e) \\ (b_i) \end{pmatrix} = \begin{bmatrix} [S_{ee}] & [S_{ei}] \\ [S_{ie}] & [S_{ii}] \end{bmatrix} \cdot \begin{pmatrix} (a_e) \\ (a_i) \end{pmatrix} \tag{6}$$

Regarding the power-wave chain matrix expression (T-parameters), we can find several definitions in the litterature. In this paper, we keep the defintion presented by Frickey [1] in order to expend his paper to multiport analysis.

$$\begin{pmatrix}(a_e)\\(b_e)\end{pmatrix} = \begin{bmatrix}[T_{ee}] & [T_{ei}]\\ [T_{ie}] & [T_{ii}]\end{bmatrix}.\begin{pmatrix}(b_i)\\(a_i)\end{pmatrix} \tag{7}$$

III. Multiport Parameters Conversions

By definition, the well known relation between impedance and admittance remains in multiport analysis as expressed in (8):

$$[Y] = [Z]^{-1} \tag{8}$$

Conversions between electrical parameters are deduced from algebraic manipulation from equations (2) to (5).

$$[Z] = \begin{bmatrix}[A].[C]^{-1} & [A].[C]^{-1}.[D]-[B]\\ [C]^{-1} & [C]^{-1}.[D]\end{bmatrix} \tag{9}$$

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix} = \begin{bmatrix}[Z_{ee}].[Z_{ie}]^{-1} & [Z_{ee}].[Z_{ie}]^{-1}.[Z_{ii}]-[Z_{ei}]\\ [Z_{ie}]^{-1} & [Z_{ie}]^{-1}.[Z_{ii}]\end{bmatrix} \tag{10}$$

$$[h] = \begin{bmatrix}[Z_{ee}]-[Z_{ei}].[Z_{ii}]^{-1}.[Z_{ie}] & [Z_{ei}].[Z_{ii}]^{-1}\\ -[Z_{ii}]^{-1}.[Z_{ie}] & [Z_{ii}]^{-1}\end{bmatrix} \tag{11}$$

$$[Z] = \begin{bmatrix}[h_{ee}]-[h_{ei}].[h_{ii}]^{-1}.[h_{ie}] & [h_{ei}].[h_{ii}]^{-1}\\ -[h_{ii}]^{-1}.[h_{ie}] & [h_{ii}]^{-1}\end{bmatrix} \tag{12}$$

$$[Y] = \begin{bmatrix}[D].[B]^{-1} & [C]-[D].[B]^{-1}.[A]\\ -[B]^{-1} & [B]^{-1}.[A]\end{bmatrix} \tag{13}$$

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix} = \begin{bmatrix}-[Y_{ie}]^{-1}.[Y_{ii}] & -[Y_{ie}]^{-1}\\ [Y_{ei}]-[Y_{ee}].[Y_{ie}]^{-1}.[Y_{ii}] & -[Y_{ie}]^{-1}.[Y_{ee}]\end{bmatrix} \tag{14}$$

$$[h] = \begin{bmatrix}[Y_{ee}]^{-1} & -[Y_{ee}]^{-1}.[Y_{ei}]\\ [Y_{ie}].[Y_{ee}]^{-1} & [Y_{ii}]-[Y_{ie}].[Y_{ee}]^{-1}.[Y_{ei}]\end{bmatrix} \tag{15}$$

$$[Y] = \begin{bmatrix}[h_{ee}]^{-1} & -[h_{ee}]^{-1}.[h_{ei}]\\ [h_{ie}].[h_{ee}]^{-1} & [h_{ii}]-[h_{ie}].[h_{ee}]^{-1}.[h_{ei}]\end{bmatrix} \tag{16}$$

Substituting Kurokawa's power-waves (1) in (6) leads us to $[S]$ to $[Z]$ and $[Y]$ matrices conversions:

$$[Z] = [G_0]^{-1}.([I]-[S])^{-1}.([S].[Z_0]+[Z_0^*]).[G_0] \tag{17}$$

$$[S] = [G_0].([Z]-[Z_0^*]).([Z]+[Z_0])^{-1}.[G_0]^{-1} \tag{18}$$

$$[Y] = [G_0]^{-1}.([S].[Z_0]+[Z_0^*])^{-1}.([I]-[S]).[G_0] \tag{19}$$

$$[S] = [G_0].([I]-[Z_0^*].[Y]).([I]+[Z_0].[Y])^{-1}.[G_0]^{-1} \tag{20}$$

with

$$[G_0] = \mathrm{diag}\{g_1,\ldots,g_n,\ldots,g_N\} \tag{21}$$

$$[Z_0] = \mathrm{diag}\{Z_1,\ldots,Z_n,\ldots,Z_N\} \tag{22}$$

and $[I]$ is the identity matrix. $[G_0]$ and $[Z_0]$ are diagonal matrices (terms outside the diagonal are zero) where each term is related to a port reference impedance Z_n and

$$g_n = \frac{1}{\sqrt{|\Re\{Z_n\}|}} \tag{23}$$

Multiport transfert parameter matrix ($[T]$), useful to cascade multiport blocks as detailed in [6], is related to the [S] matrix as follow:

$$[T] = \begin{bmatrix}[S_{ie}]^{-1} & -[S_{ie}]^{-1}.[S_{ii}]\\ [S_{ee}].[S_{ie}]^{-1} & [S_{ei}]-[S_{ee}].[S_{ie}]^{-1}.[S_{ii}]\end{bmatrix} \tag{24}$$

$$[S] = \begin{bmatrix}[T_{ie}].[T_{ee}]^{-1} & [T_{ii}]-[T_{ie}].[T_{ee}]^{-1}.[T_{ei}]\\ [T_{ee}]^{-1} & -[T_{ee}]^{-1}.T_{ei}\end{bmatrix} \tag{25}$$

Notice that the chain matrices $[ABCD]$ and $[T]$ are properly defined only when the system is balanced (same number of internal and external ports). If the system is unbalanced, we have to ensure the uniqueness of the solution and then to apply the pseudo-inverse operator instead of the inverse matrix function.

IV. Multiport S-parameters Normalization

Considering the reference impedance matrix (22) and the conversions (17) and (18), we can express a change of reference impedance of a multiport S-parameter $[S]$ from $[Z_0]$ to $[Z_0']$ as follow :

$$[S'] = [A]^{-1}.([S]-[\rho^*]).([I]-[\rho].[S])^{-1}.[A^*] \tag{26}$$

where

$$[A] = [G_0']^{-1}.[G_0].[[I]-[\rho^*]] \tag{27}$$

$$[\rho] = [[Z_0']-[Z_0]].[[Z_0']+[Z_0^*]]^{-1} \tag{28}$$

$[I]$ is the identity matrix, $[G_0]$, $[G_0']$, $[Z_0]$ and $[Z_0']$ are defined in (21) and (22) respectively. All matrices, except $[S]$ and $[S']$, are diagonals.

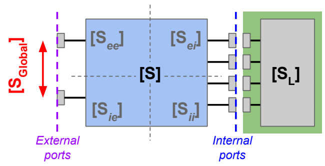

Fig. 2. Illustration of an unbalanced multiport S-parameter terminated with a multiport load at its internals ports.

V. Multiport Embedding and SnP reduction

As depicted by figure 2, the embedding procedure consists on calculating the S-parameters at the external ports according to the perfectly known S-parameters connected at the internal ports. The formula already demonstrated in [7], and eventually in the annex of [8], is:

$$[S_{Global}] = [S_{ee}]+[S_{ei}].([I]-[S_L].[S_{ii}])^{-1}.[S_L].[S_{ie}] \tag{29}$$

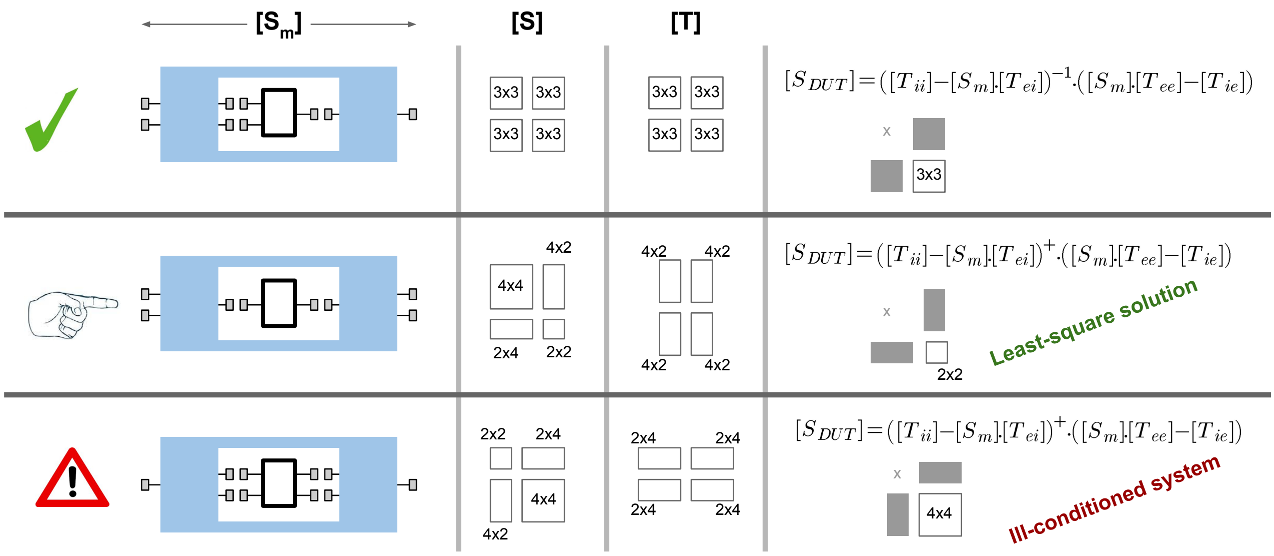

Fig. 3. Illustration of de-embedding with multiport T-matrices. The first line represents a balanced system. The second line is an unbalanced system with a number of external ports higher than internal ones ($N_e > N_i$). The matrix inversion has to be done with the pseudo-inverse operator ($\bullet^+$). The third line represents an unbalanced system where $N_i > N_e$. There is not uniqueness of the solution for the matrix inversion. S_{DUT} solutions are presented here. More cascading properties of the multiport T-matrix are available in [6]

We can apply the same demonstration to $[Z]$ or $[Y]$ matrices to obtain those embedding results:

$$[Z_{Global}] = [Z_{ee}] - [Z_{ei}].([Z_{ii}] + [Z_L])^{-1}.[Z_{ie}] \tag{30}$$

$$[Y_{Global}] = [Y_{ee}] - [Y_{ei}].([Y_{ii}] + [Y_L])^{-1}.[Y_{ie}] \tag{31}$$

We can notice that the embedding procedure is a reduction of the number of available port on a $[S]$ matrix when we know the terminations applied to the other ports.

VI. De-embedding Problems

According to figure 2, the de-embedding procedure consists on extracting the S-parameters at the internal ports ($S_L = S_{int}$) when $[S]$ and $S_{Global} = S_{ext}$ are known. We can calculate from equation (29):

$$\begin{aligned} S_{int} =& S_{ei}{}^{-1}.(S_{ext} - S_{ee}) \\ & .\left(S_{ie} + S_{ii}.S_{ei}{}^{-1}.(S_{ext} - S_{ee})\right)^{-1} \end{aligned} \tag{32}$$

Equation (32) works only when $[S]$ describe a balanced system and is similar to 8-error term model multiport VNA calibration [9]. If the number of internal ports (N_i) is lower than the number of external port (N_e), we can use the pseudo-inverse operator to invert S_{ei} and get the least-square solution. Otherwise, the solution can not be solved as it is and requires more assumptions. This perfectly illustrates that S to T matrix transformation is lossy with unbalanced networks when $N_i > N_e$ as depicted by figure 3. Figure 3 presents a possible way to extract S_{DUT} from the T-parameters.

VII. Conclusion

This paper presents the equations for converting electrical parameters matrix representations. Equations are derived from the definitions provided by Frickey in 1994 for 2-port parameters [1] but are, here, extended to multiport analysis. According to the formulas presented in this paper, the reader can handle, convert, normalize or reduce any matrix representing a linear multiport.

References

[1] D. A. Frickey, "Conversions between S, Z, Y, H, ABCD, and T parameters which are valid for complex source and load impedances," *IEEE Transactions on Microwave Theory and Techniques*, vol. 42, no. 2, pp. 205–211, February 1994.

[2] R. B. Marks, D. F. Williams, and D. A. Frickey, "Comments on "conversions between s, z, y, h, abcd, and t parameters which are valid for complex source and load impedances" [with reply]," *IEEE Transactions on Microwave Theory and Techniques*, vol. 43, no. 4, pp. 914–915, April 1995.

[3] J. Tippet and R. Speciale, "A Rigorous Technique for Measuring the Scattering Matrix of a Multiport Device with a 2-Port Network Analyzer," *IEEE Transactions on Microwave Theory and Techniques*, vol. 30, no. 5, pp. 661–666, may 1982.

[4] J. A. Dobrowolski, *Microwave network design using the scattering matrix*. Artech House, 2010.

[5] K. Kurokawa, "Power Waves and the Scattering Matrix," *IEEE Transactions on Microwave Theory and Techniques*, vol. 13, no. 2, pp. 194–202, March 1965.

[6] J. Frei, X.-D. Xiao-Ding Cai, and S. Muller, "Multiport S-Parameter and T-Parameter Conversion With Symmetry Extension," *IEEE Transactions on Microwave Theory and Techniques*, vol. 56, no. 11, pp. 2493–2504, nov 2008.

[7] T. T. Ha, *Solid-state microwave amplifier design*. New York, Wiley-Interscience, 1981.

[8] A. Ferrero, U. Pisani, and K. Kerwin, "A new implementation of a multiport automatic network analyzer," *IEEE Transactions on Microwave Theory and Techniques*, vol. 40, no. 11, pp. 2078–2085, 1992.

[9] A. Ferrero, F. Sampietro, and U. Pisani, "Multiport vector network analyzer calibration: a general formulation," *IEEE Transactions on Microwave Theory and Techniques*, vol. 42, no. 12, pp. 2455–2461, 1994.